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TOPIC - Resolution into Factors (Trigonometry)

Introduction → In this chapter we will present an elementary Treatment of the Topic Resolution of Trigonometrical Functions into Factors and also Resolving x^n-1 and x^n+1 into factors according as n is an even or odd.

1. To Express $\sin\theta$ as a product of an infinite series of factors.

Proof:→ We have
$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$
$$= 2\sin\frac{\theta}{2}\sin\left(\pi/2+\frac{\theta}{2}\right)$$
$$= 2\sin\frac{\theta}{2}\sin\left(\frac{\pi+\theta}{2}\right) \quad \text{——— (I)}$$

Similarly, Replacing θ by $\frac{\theta}{2}$ and $\frac{\pi}{2}+\frac{\theta}{2}$ successively in (I) we get
$$\sin\frac{\theta}{2} = 2\sin\frac{\theta}{2^2}\sin\left(\frac{\pi}{2}+\frac{\theta}{2^2}\right) = 2\sin\frac{\theta}{2^2}\sin\left(\frac{2\pi}{2^2}+\frac{\theta}{2^2}\right) \text{—(II)}$$
and
$$\sin\left(\frac{\pi}{2}+\frac{\theta}{2}\right) = 2\sin\frac{1}{2}\left(\frac{\pi}{2}+\frac{\theta}{2}\right).\sin\left\{\frac{\pi}{2}+\frac{1}{2}\left(\frac{\pi}{2}+\frac{\theta}{2}\right)\right\}$$
$$= 2\sin\left(\frac{\pi}{2^2}+\frac{\theta}{2^2}\right)\sin\left(\frac{\pi}{2}+\frac{\pi}{2^2}+\frac{\theta}{2^2}\right)$$
$$= 2\sin\left(\frac{\pi}{2^2}+\frac{\theta}{2^2}\right)\sin\left(\frac{2\pi}{2^2}+\frac{\pi}{2^2}+\frac{\theta}{2^2}\right)$$
$$= 2\sin\left(\frac{\pi}{2^2}+\frac{\theta}{2^2}\right)\sin\left(\frac{3\pi}{2^2}+\frac{\theta}{2^2}\right) \text{——— (III)}$$

On putting the values of $\sin\frac{\theta}{2}$ and $\sin\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$ From (II) and (III) in R.H.S of (I), we get
$$\sin\theta = 2\cdot 2\sin\frac{\theta}{2^2}\sin\left(\frac{2\pi}{2^2}+\frac{\theta}{2^2}\right).2\sin\left(\frac{\pi}{2^2}+\frac{\theta}{2^2}\right).\sin\left(\frac{3\pi}{2^2}+\frac{\theta}{2^2}\right)$$
$$= 2^3\sin\frac{\theta}{2^2}\sin\frac{\pi+\theta}{2^2}\sin\frac{2\pi+\theta}{2^2}\sin\frac{3\pi+\theta}{2^2}$$

Again decomposing each of the above sine as in (I), we get
$$\sin\theta = 2^3.2^4\sin\frac{\theta}{2^3}\sin\frac{\pi+\theta}{2^3}\sin\frac{2\pi+\theta}{2^3}.\sin\frac{3\pi+\theta}{2^3}.\sin\frac{4\pi+\theta}{2^3}\sin\frac{5\pi+\theta}{2^3}$$
$$.\sin\frac{6\pi+\theta}{2^3}.\sin\frac{7\pi+\theta}{2^3}$$
$$= 2^7\sin\frac{\theta}{2^3}\sin\frac{\pi+\theta}{2^3}\sin\frac{2\pi+\theta}{2^3}\sin\frac{3\pi+\theta}{2^3}\sin\frac{4\pi+\theta}{2^3}\sin\frac{5\pi+\theta}{2^3}\sin\frac{6\pi+\theta}{2^3}\sin\frac{7\pi+\theta}{2^3}$$

i.e. $\sin\theta = 2^{(2^3-1)}\sin\frac{\theta}{2^3}\sin\frac{\pi+\theta}{2^3}\sin\frac{2\pi+\theta}{2^3}\sin\frac{3\pi+\theta}{2^3}\sin\frac{4\pi+\theta}{2^3}\sin\frac{5\pi+\theta}{2^3}\sin\frac{6\pi+\theta}{2^3}\sin\frac{7\pi+\theta}{2^3}$

$= 2^{(2^3-1)}\sin\frac{\theta}{2^3}\sin\frac{\pi+\theta}{2^3}\sin\frac{2\pi+\theta}{2^3}\sin\frac{3\pi+\theta}{2^3}\sin\frac{4\pi+\theta}{2^3}\sin\frac{5\pi+\theta}{2}\sin\frac{6\pi+\theta}{2^3}\sin\frac{(2^3-1)\pi+\theta}{2^3}$

Repeating the same process n times, we get -

$\sin\theta = 2^{(2^n-1)}\sin\frac{\theta}{2^n}\sin\frac{\pi+\theta}{2^n}\sin\frac{2\pi+\theta}{2^n}\sin\frac{3\pi+\theta}{2^n}\cdots\sin\frac{(2^n-2)\pi+\theta}{2^n}\sin\frac{(2^n-1)\pi+\theta}{2^n}$

say $2^n = P$, P is a power of 2.

$\sin\theta = 2^{(P-1)}\sin\frac{\theta}{P}\sin\frac{\pi+\theta}{P}\sin\frac{2\pi+\theta}{P}\sin\frac{3\pi+\theta}{P}\cdots\sin\frac{(P-2)\pi+\theta}{P}\sin\frac{(P-1)\pi+\theta}{P}$ ——— (IV)

the last factor in (IV) is

$\sin\frac{(P-1)\pi+\theta}{P} = \sin\left[\pi - \frac{\pi-\theta}{P}\right] = \sin\left(\frac{\pi-\theta}{P}\right)$

the last but one factor in (IV) is

$\sin\frac{(P-2)\pi+\theta}{P} = \sin\left[\pi - \frac{2\pi-\theta}{P}\right] = \sin\left(\frac{2\pi-\theta}{P}\right)$

[using $\sin(\pi-\theta) = \sin\theta$]

Now the total no of factors is $2^n = P$ which is an even no.

thus total no of factors in (IV) is even

we now leave the first factor and combine the other factors as follows

the 2nd factor with the last factor,
the 3rd factor with the last but one
the fourth factor with the last but two and so on.

In this way, we have made the combination of two factors out of the remaining $(P-1)$ factors as we have left the first term. obviously $P-1$ is an odd and after making groups of two we shall left with one factor which will be the $\left(\frac{P}{2}+1\right)$th factor from the begining which can not be combined with any other factor, and its value is

$\sin\frac{\left(\frac{P}{2}+1-1\right)\pi+\theta}{P} = \sin\frac{\left(\frac{P\pi}{2}+\theta\right)}{P} = \sin\left(\frac{\pi}{2}+\frac{\theta}{P}\right) = \cos\frac{\theta}{P}$

Thus after combining as explained above equation (10) reduces to

$$\sin\theta = 2^{p-1}\sin\frac{\theta}{p}\cdot\left\{\sin\frac{\pi+\theta}{p}\cdot\sin\frac{\pi-\theta}{p}\right\}\left\{\sin\frac{2\pi+\theta}{p}\cdot\sin\frac{2\pi-\theta}{p}\right\}\cdots$$

$$\cdots\times\left\{\frac{\sin\left(\frac{p}{2}+1\right)\pi+\theta}{p}\cdot\frac{\sin\left(\frac{p}{2}-1\right)\pi+\theta}{p}\right\}\cdot\cos\frac{\theta}{p}$$

$$=2^{p-1}\cdot\sin\frac{\theta}{p}\left\{\sin^2\frac{\pi}{p}-\sin^2\frac{\theta}{p}\right\}\left\{\sin^2\frac{2\pi}{p}-\sin^2\frac{\theta}{p}\right\}\cdots$$

$$\cdots\left\{\sin^2\frac{\left(\frac{p}{2}-1\right)\pi}{p}-\sin^2\frac{\theta}{p}\right\}\cdot\cos\frac{\theta}{p} \quad \text{——— (V)}$$

[using $\sin(A+B)\cdot\sin(A-B)=\sin^2A-\sin^2B$]

$$\Rightarrow \frac{\sin\theta}{\sin\frac{\theta}{p}} = 2^{p-1}\left\{\sin^2\frac{\pi}{p}-\sin^2\frac{\theta}{p}\right\}\left\{\sin^2\frac{2\pi}{p}-\sin^2\frac{\theta}{p}\right\}\cdots\left(\sin^2\frac{\left(\frac{p}{2}-1\right)\pi}{p}-\sin^2\frac{\theta}{p}\right)\cos\frac{\theta}{p} \quad \text{——— (VI)}$$

Now the expression on the R.H.S. is continuous up to the values $\theta=0$

Hence make $\theta\to0$, therefore we have

$$\lim_{\theta\to0}\frac{\sin\theta}{\theta}=1 \quad \lim_{\theta\to0}\sin^2\frac{\theta}{p}\to0 \quad \lim_{\theta\to0}\cos\frac{\theta}{p}\to1$$

and $$\lim_{\theta\to0}\frac{\sin\theta}{\sin\frac{\theta}{p}} = \lim_{\theta\to0}\left(\frac{\sin\theta}{\theta}\right)\times\frac{\theta\times1}{\lim_{\theta\to0}\frac{\sin\frac{\theta}{p}}{\theta/p}}\times\frac{1}{\theta/p}=p$$

therefore, from (V)

$$p = 2^{p-1}\sin^2\frac{\pi}{p}\,\sin^2\frac{2\pi}{p}\,\sin^2\frac{3\pi}{p}\cdots\sin^2\frac{\left(\frac{p}{2}-1\right)\pi}{p} \quad \text{——— (VII)}$$

Now (V) $\div$ (VII)

we have

$$\frac{\sin\theta}{p} = \sin\frac{\theta}{p}\left\{1-\frac{\sin^2\frac{\theta}{p}}{\sin^2\frac{\pi}{p}}\right\}\left\{1-\frac{\sin^2\frac{\theta}{p}}{\sin^2\frac{2\pi}{p}}\right\}\left\{1-\frac{\sin^2\theta/p}{\sin^2\frac{3\pi}{p}}\right\}\cdots$$

$$\cdots\left\{1-\frac{\sin^2\frac{\theta}{p}}{\sin^2\left(\frac{p}{2}-1\right)\pi}\right\}\cos\frac{\theta}{p} \quad \text{——— (VIII)}$$

$$\Rightarrow \sin\theta = p\sin\frac{\theta}{p}\cos\frac{\theta}{p}\prod_{r=1}^{\frac{p-2}{2}}\left\{1-\frac{\sin^2\theta/p}{\sin^2\frac{r\pi}{p}}\right\} \quad \text{——— (IX)}$$

Now let $p\to\infty$ so that and $\lim_{p\to\infty}\cos\frac{\theta}{p}$.

$$\lim_{p\to\infty}p\sin\frac{\theta}{p} = \lim_{p\to\infty}\frac{\sin\frac{\theta}{p}}{\theta/p}\times\theta = \theta$$ put $1/p=y$ when $p\to\infty$ $y\to0$

Therefore $\lim_{p\to\infty}\left(p\sin\frac{\theta}{p}\right) = \lim_{y\to 0}\left(\frac{\sin y\theta}{y\theta}\right)\times\theta = \theta.$

and $\lim_{p\to\infty}\cos\frac{\theta}{p} = \lim_{y\to 0}\cos y\theta = 1$

Also $$\lim_{p\to\infty}\frac{\sin^2\theta/p}{\sin^2\frac{\pi}{p}} = \lim_{p\to\infty}\frac{\left\{\frac{\sin^2\theta/p}{\frac{\theta^2}{p^2}}\times\frac{\theta^2}{p^2}\right\}}{\frac{\sin^2\frac{\pi}{p}}{\frac{\pi^2}{p^2}}\times\frac{\pi^2}{p^2}} = \frac{\theta^2}{\pi^2}, \text{ (As above)}$$

Similarly $\lim_{p\to\infty}\frac{\sin^2\theta/p}{\sin^2\frac{2\pi}{p}} = \frac{\theta^2}{2^2\pi^2}$, $\lim_{p\to\infty}\frac{\sin^2\theta/p}{\sin^2\frac{3\pi}{p}} = \frac{\theta^2}{3^2\pi^2}$ - - - - -

and $\lim_{p\to\infty}\frac{\sin^2\theta/p}{\sin^2\frac{r\pi}{p}} = \frac{\theta^2}{r^2\pi^2}$

Thus from Equation (viii) and (ix), we have.

$$\sin\theta = \theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right) - - - - \infty$$

$$\Rightarrow \sin\theta = \theta\prod_{r=1}^{r=\infty}\left(1-\frac{\theta^2}{r^2\pi^2}\right)$$